MTB

<u>NOTATIONS :</u>

$\mathbb{Z}$ = Set of integers.

$\mathbb{N}$ = Set of natural numbers = $\{n \in \mathbb{Z} : n \geq 1\}$.

$\mathbb{Q}$ = Set of rationals.

$\mathbb{R}$ = Set of real numbers.

$\mathbb{C}$ = Set of complex numbers.

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1. Let $A = \mathbb{Z}[\omega]$ be the subring of complex numbers where ω is a non-real 6th root of unity. In the polynomial ring $A[x]$, let I be the ideal generated by $(x^2 + x + 1)$. Prove or disprove that $A[x]/I$ is an integral domain.

2. For a ring R, let R^* denote the set of units of R. Are the pair of groups isomorphic?

   (a) $(\mathbb{C}^*, \cdot)$, $((\mathbb{R}^2)^*, \cdot)$

   (b) $(\mathbb{R}, +)$, $(\mathbb{C}, +)$

   Justify your answer.

3. Let $R = \mathbb{C}[X, Y, Z]/(2X^2Y - 3XY^3 + Z)$. Show that R is a unique factorization domain.

4. Let F be a field of q elements and n a natural number. Compute the order of $SL_n(F)$. Here $SL_n(F)$ denotes the special linear group of degree n i.e., $n \times n$ matrices over F with determinant 1.

5. Let V be a vector space over $\mathbb{C}$, and let T be an invertible linear operator on V. Prove that there exists a polynomial $p \in \mathbb{C}[z]$ such that $T^{-1} = p(T)$.

6. Let V be a finite-dimensional vector space over $\mathbb{C}$ or $\mathbb{R}$ and let T be a linear operator on V. Prove that $T^2 = T$ if and only if

$$\operatorname{rank}(T) + \operatorname{rank}(I - T) = \dim V.$$

7. Let $\mathcal{H}$ be a Hilbert space, and let $\dim \mathcal{H} > 1$. Let $\{v_1, v_2\} \subseteq \mathcal{H}$ be an orthogonal set of nonzero vectors. Suppose

$$Tx = \langle x, v_1 \rangle v_2 + \langle x, v_2 \rangle v_1 \qquad (\forall x \in \mathcal{H}).$$

Prove that T is a bounded linear operator. Also compute the norm of the operator T.

8. Let F be a finite-dimensional subspace and C a closed subspace of a Banach space B. Prove that $\{f + g : f \in F, g \in C\}$ is a closed subspace of B.

9. Let $X_i, i \geq 1$ be i.i.d. discrete random variables with mean μ and variance σ^2. Let $k > 1$. Define the sequence

$$Y_n := \frac{X_1 X_2 \ldots X_k + X_2 X_3 \ldots X_{k+1} + \ldots + X_{n-k+1} X_{n-k+2} \ldots X_n}{n}.$$

Find $\lim_{n\to\infty} \mathbb{E}[Y_n]$ and $\lim_{n\to\infty} n^{3/4} \mathbb{E}[(Y_n - \mathbb{E}[Y_n])^2]$ where $\mathbb{E}$ denotes the expectation.

10. Let n and m be integers such that 5 divides $1 + 2n^2 + 3m^2$. Show that 5 divides $n^2 - 1$.